

On Study K^h Generalized Birecurrent Affinely Connected Space

Fahmi Yaseen Abdo Qasem^{a*}, Amani Mohammed Abdulla Hanballa^b

^aDepartment of Mathematics, Faculty of Education-Aden, University of Aden, Khormaksar, Aden, Yemen

^bDepartment of Mathematics, Community College, Dar-Saad, Aden, Yemen

^{a,b}Email: Fahmiyassen1@gmail.com, ahanballa@yahoo.com

Abstract

In the present paper we introduce a K^h – generalized birecurrent space which characterized by the condition $K_{jkh|\ell|m}^i = \lambda_\ell K_{jkh|m}^i + b_{\ell m} K_{jkh}^i$, $K_{jkh}^i \neq 0$, where λ_ℓ and $b_{\ell m}$ are non-zero covariant vector fields and covariant tensor field of second order, respectively. This space satisfies the condition of affinely connected space called K^h – generalized birecurrent affinely connected space.

Keywords: Finsler space; K^h – Generalized birecurrent space; Ricci tensor.

1. Introduction

H. D. Pande and B. Single [4] discussed the recurrence property in an affinely connected space. P. K. Dwivedi [7] worked out the role of P^* – reducible space in affinely connected space. A. A. M. Saleem [2] obtained some results when the C^h – generalized birecurrent and C^h – special generalized birecurrent are affinely connected spaces. A. A. A. Muhib [1] obtained some results when R^h – generalized trirecurrent and R^h – special generalized trirecurrent are affinely connected spaces. M. A. A. Ali [5] obtained certain identities in a K^h – birecurrent affinely connected spaces. N. S. H. Hussien [6] obtained certain identities in a K^h – recurrent affinely connected spaces.

* Corresponding author.

Let F_n be an n -dimensional Finsler space equipped with the metric function a $F(x, y)$ satisfying the request conditions [3].

The vectors y_i , y^i and the metric tensor g_{ij} satisfies the following relations

$$(1.1) \quad a) \quad y^i_{|k} = 0 \quad \text{and} \quad b) \quad g_{ij|k} = 0,$$

The h – covariant differentiation with respect to x^k , commute with the partial differentiation with respect to y^j according to

$$(1.2) \quad a) \quad \partial_j(X^i_{|k}) - (\partial_j X^i)_{|k} = X^r(\partial_j \Gamma^i_{rk}) - (\partial_r X^i) P^r_{jk},$$

where

$$(1.2) \quad b) \quad P^r_{jk} := (\partial_j \Gamma^{*r}_{hk}) y^h = \Gamma^{*r}_{jhk} y^h$$

The tensor K^i_{rkh} is called *Cartan's fourth curvature tensor* which is skew-symmetric in its last two lower indices k and h , i.e.

$$(1.3) \quad K^i_{jkh} = -K^i_{jhk}.$$

The associate tensor K_{ijkh} of the curvature tensor K^i_{jkh} is given by

$$(1.4) \quad K_{ijkh} := g_{rj} K^r_{ikh}.$$

The Ricci tensor K_{jk} of the curvature tensor K^i_{jkh} is given by

$$(1.5) \quad K^i_{jki} = K_{jk}.$$

The curvature tensor K^i_{jkh} satisfies the following relations too

$$(1.6) \quad K^i_{jkh} y^j = H^i_{kh}$$

and

$$(1.7) \quad H^i_{jkh} - K^i_{jkh} = P^i_{jk|h} + P^r_{jk} P^i_{rh} - h/k.$$

Berwald curvature tensor H^i_{jkh} satisfies the relation

$$(1.8) \quad H^i_{jkh} y^j = H^i_{kh}$$

and

$$(1.9) \quad H_{jkh}^i = \dot{\partial}_j H_{kh}^i,$$

where H_{kh}^i called $h(v)$ – torsion tensor.

Also, satisfies bianchi identity

$$(1.10) \quad a) H_{jkh}^i + H_{hjk}^i + H_{khj}^i = 0$$

and it is skew-symmetric in its last two lower indices, i.e.

$$(1.10) \quad b) H_{jkh}^i = -H_{jhk}^i.$$

The deviation tensor H_k^i is positively homogeneous of degree two in y^i and satisfies

$$(1.11) \quad H_{hk}^i y^h = H_k^i,$$

$$(1.12) \quad H_{jk} = H_{jki}^i,$$

$$(1.13) \quad H_k = H_{ki}^i,$$

and

$$(1.14) \quad H = \frac{1}{n-1} H_i^i,$$

where H_{jk} and H are called *h-Ricci tensor* and *curvature scalar*, respectively. Since contraction of the indices does not affect the homogeneity in y^i , hence the tensors H_{rk} , H_r and the scalar H are also homogeneous of degree zero, one and two in y^i , respectively.

The associate tensor H_{ijkh} of Berwald curvature tensor H_{jkh}^i is given by

$$(1.15) \quad H_{ijkh} = g_{rj} H_{ikh}^r.$$

The contraction of the indices i and j in (1.10a) and by using (1.12) and the skew-symmetric property of the curvature tensor H_{jkh}^i in the last two lower indices, shows that the h – Ricci tensor satisfies

$$(1.16) \quad H_{rkh}^r = H_{hk} - H_{kh}.$$

The tensor $H_{jh.k}$ defined by

$$(1.17) \quad H_{jh.k} := g_{ih} H_{jk}^i.$$

Cartan's fourth curvature tensor K_{jkh}^i satisfies the following identity known as *Bianchi identity*

$$(1.18) \quad K_{jkh|\ell}^i + K_{j\ell k|h}^i + K_{jh\ell|k}^i + y^r \{ (\partial_s \Gamma_{jk}^{*i}) K_{r h \ell}^s + (\partial_s \Gamma_{j\ell}^{*i}) K_{r k h}^s + (\partial_s \Gamma_{jh}^{*i}) K_{r \ell k}^s \} = 0.$$

A Finsler space whose Berwald's connection parameter G_{jk}^i is independent of y^i is called an *affinely connected space* (Berwald space). Thus, an affinely connected space has some properties as follows:

$$(1.19) \quad G_{jkh}^i = 0$$

and

$$(1.20) \quad C_{ijk|h} = 0.$$

The connection parameters Γ_{jk}^{*i} of Cartan and G_{jk}^i of Berwald coincide in an affinely connected space and they are independent of the direction arguments [3], i.e.

$$(1.21) \quad G_{jkh}^i = \partial_j G_{kh}^i = 0$$

and

$$(1.22) \quad \partial_j \Gamma_{kh}^{*i} = 0.$$

N. S. H. Hussein [3] introduce the K^h -recurrent space which characterized by the condition

$$(1.23) \quad K_{jkh|\ell}^i = \lambda_\ell K_{jkh}^i, \quad K_{jkh}^i \neq 0,$$

where the covariant vector field λ_ℓ being the recurrence vector field.

2. An K^h – Generalized Birecurrent Space

Let us consider a Finsler space F_n whose Cartan's fourth curvature tensor K_{jkh}^i satisfies the condition

$$(2.1) \quad K_{jkh|\ell|m}^i = \lambda_\ell K_{jkh|m}^i + b_{\ell m} K_{jkh}^i, \quad K_{jkh}^i \neq 0,$$

where λ_ℓ and $b_{\ell m}$ are non-zero covariant vector field and covariant tensor field of second order, respectively. The space satisfying the condition (2.1) will be called K^h -generalized birecurrent space. We shall denote it briefly by K^h -GBR- F_n .

Transvecting (2.1) by y^j , using (1.1a) and (1.6), we get

$$(2.2) \quad H_{kh|\ell|m}^i = \lambda_\ell H_{kh|m}^i + b_{\ell m} H_{kh}^i.$$

Theorem 2.1. In $K^h-GBR-F_n$, the $h(v)$ -torsion tensor H_{kh}^i is generalized birecurrent.

Differentiating (2.2) partially with respect to y^j and using (1.9), we get

$$(2.3) \quad \partial_j H_{kh|\ell|m}^i = (\partial_j \lambda_\ell) H_{kh|m}^i + \lambda_\ell \partial_j (H_{kh|m}^i) + (\partial_j b_{\ell m}) H_{kh}^i + b_{\ell m} H_{jkh}^i.$$

Using the commutation formula exhibited by (1.2a) for $(H_{kh|\ell}^i)$ and (H_{kh}^i) in (2.3) and using (1.9), we get

$$(2.4) \quad \{ \partial_j (H_{kh|\ell}^i) \}_{|m} + H_{kh|\ell}^r (\partial_j \Gamma_{rm}^{*i}) - H_{rh|\ell}^i (\partial_j \Gamma_{km}^{*r}) - H_{kr|\ell}^i (\partial_j \Gamma_{hm}^{*r}) \\ - H_{kh|\ell}^i (\partial_j \Gamma_{\ell m}^{*r}) - \partial_r (H_{kh|\ell}^i) P_{jm}^r = (\partial_j \lambda_\ell) H_{kh|m}^i + \lambda_\ell H_{jkh|m}^i + \\ \lambda_\ell [H_{kh}^r (\partial_j \Gamma_{rm}^{*i}) - H_{rh}^i (\partial_j \Gamma_{km}^{*r}) - H_{kr}^i (\partial_j \Gamma_{hm}^{*r}) - H_{rkh}^i P_{jm}^r] + \\ (\partial_j b_{\ell m}) H_{kh}^i + b_{\ell m} H_{jkh}^i.$$

Again applying the commutation formula exhibited by (1.2a) for (H_{kh}^i) in (2.4) and using (1.9), we get

$$\{ H_{jkh|\ell}^i + H_{kh}^r (\partial_j \Gamma_{r\ell}^{*i}) - H_{rh}^i (\partial_j \Gamma_{k\ell}^{*r}) - H_{kr}^i (\partial_j \Gamma_{h\ell}^{*r}) - H_{rkh}^i P_{j\ell}^r \}_{|m} \\ + H_{kh|\ell}^r (\partial_j \Gamma_{rm}^{*i}) - H_{rh|\ell}^i (\partial_j \Gamma_{km}^{*r}) - H_{kr|\ell}^i (\partial_j \Gamma_{hm}^{*r}) - H_{kh|\ell}^i (\partial_j \Gamma_{\ell m}^{*r}) - \\ \{ H_{rkh|\ell}^i + H_{kh}^s (\partial_r \Gamma_{s\ell}^{*i}) - H_{sh}^i (\partial_r \Gamma_{k\ell}^{*s}) - H_{ks}^i (\partial_r \Gamma_{h\ell}^{*s}) - H_{skh}^i P_{r\ell}^s \} P_{jm}^r \\ = (\partial_j \lambda_\ell) H_{kh|m}^i + \lambda_\ell H_{jkh|m}^i + \lambda_\ell [H_{kh}^r (\partial_j \Gamma_{rm}^{*i}) - H_{rh}^i (\partial_j \Gamma_{km}^{*r}) \\ - H_{kr}^i (\partial_j \Gamma_{hm}^{*r}) - H_{rkh}^i P_{jm}^r] + (\partial_j b_{\ell m}) H_{kh}^i + b_{\ell m} H_{jkh}^i$$

which can be written as

$$(2.5) \quad H_{jkh|\ell|m}^i + \{ H_{kh}^r (\partial_j \Gamma_{r\ell}^{*i}) - H_{rh}^i (\partial_j \Gamma_{k\ell}^{*r}) - H_{kr}^i (\partial_j \Gamma_{h\ell}^{*r}) - H_{rkh}^i P_{j\ell}^r \}_{|m} + \\ H_{kh|\ell}^r (\partial_j \Gamma_{rm}^{*i}) - H_{rh|\ell}^i (\partial_j \Gamma_{km}^{*r}) - H_{kr|\ell}^i (\partial_j \Gamma_{hm}^{*r}) - H_{kh|\ell}^i (\partial_j \Gamma_{\ell m}^{*r}) - H_{rkh|\ell}^i P_{jm}^r \\ - H_{kh}^s (\partial_r \Gamma_{s\ell}^{*i}) P_{jm}^r + H_{sh}^i (\partial_r \Gamma_{k\ell}^{*s}) P_{jm}^r + H_{ks}^i (\partial_r \Gamma_{h\ell}^{*s}) P_{jm}^r + H_{skh}^i P_{r\ell}^s P_{jm}^r = \\ \lambda_\ell H_{jkh|m}^i + b_{\ell m} H_{jkh}^i + (\partial_j \lambda_\ell) H_{kh|m}^i + \lambda_\ell H_{kh}^r (\partial_j \Gamma_{rm}^{*i}) - \lambda_\ell H_{rh}^i (\partial_j \Gamma_{km}^{*r}) \\ - \lambda_\ell H_{kr}^i (\partial_j \Gamma_{hm}^{*r}) - \lambda_\ell H_{rkh}^i P_{jm}^r + (\partial_j b_{\ell m}) H_{kh}^i.$$

This shows that

$$H_{jkh|\ell|m}^i = \lambda_\ell H_{jkh|m}^i + b_{\ell m} H_{jkh}^i$$

if and only if

$$\begin{aligned} (2.6) \quad & \left\{ H_{kh}^r(\partial_j \Gamma_{r\ell}^{*i}) - H_{rh}^i(\partial_j \Gamma_{k\ell}^{*r}) - H_{kr}^i(\partial_j \Gamma_{h\ell}^{*r}) - H_{rkh}^i P_{j\ell}^r \right\}_m + H_{kh|\ell}^r(\partial_j \Gamma_{rm}^{*i}) \\ & - H_{rh|\ell}^i(\partial_j \Gamma_{km}^{*i}) - H_{kr|\ell}^i(\partial_j \Gamma_{hm}^{*i}) - H_{kh|r}^i(\partial_j \Gamma_{\ell m}^{*r}) - H_{rkh|\ell}^i P_{jm}^r - \\ & H_{kh}^s(\partial_r \Gamma_{s\ell}^{*i}) P_{jm}^r + H_{sh}^i(\partial_r \Gamma_{k\ell}^{*s}) P_{jm}^r + H_{ks}^i(\partial_r \Gamma_{h\ell}^{*s}) P_{jm}^r + H_{skh}^i P_{r\ell}^s P_{jm}^r \\ & = (\partial_j \lambda_\ell) H_{kh|m}^i + \lambda_\ell H_{kh}^r(\partial_j \Gamma_{rm}^{*i}) - \lambda_\ell H_{rh}^i(\partial_j \Gamma_{km}^{*i}) - \lambda_\ell H_{kr}^i(\partial_j \Gamma_{hm}^{*i}) \\ & - \lambda_\ell H_{rkh}^i P_{jm}^i + (\partial_j b_{\ell m}) H_{kh}^i. \end{aligned}$$

Thus, we conclude

Theorem 2.2. In K^h -GBR- F_n , Berwald curvature tensor H_{jkh}^i is generalized birecurrent if and only if (2.6) holds good.

Transvecting (2.5) by g_{ip} , using (1.1b) and (1.15), we get

$$\begin{aligned} (2.7) \quad & H_{jpkh|\ell|m} + g_{ip} \left[\left\{ H_{kh}^r(\partial_j \Gamma_{r\ell}^{*i}) - H_{rh}^i(\partial_j \Gamma_{k\ell}^{*r}) - H_{kr}^i(\partial_j \Gamma_{h\ell}^{*r}) - H_{rkh}^i P_{j\ell}^r \right\}_m \right. \\ & + H_{kh|\ell}^r(\partial_j \Gamma_{rm}^{*i}) - H_{rh|\ell}^i(\partial_j \Gamma_{km}^{*i}) - H_{kr|\ell}^i(\partial_j \Gamma_{hm}^{*i}) - H_{kh|r}^i(\partial_j \Gamma_{\ell m}^{*r}) - \\ & H_{rkh|\ell}^i P_{jm}^r - H_{kh}^s(\partial_r \Gamma_{s\ell}^{*i}) P_{jm}^r + H_{sh}^i(\partial_r \Gamma_{k\ell}^{*s}) P_{jm}^r + H_{ks}^i(\partial_r \Gamma_{h\ell}^{*s}) P_{jm}^r + \\ & \left. H_{skh}^i P_{r\ell}^s P_{jm}^r \right] = (\lambda_\ell H_{jpkh|m} + b_{\ell m} H_{jpkh}) + g_{ip} \{ (\partial_j \lambda_\ell) H_{kh|m}^i + \\ & \lambda_\ell H_{kh}^r(\partial_j \Gamma_{rm}^{*i}) - \lambda_\ell H_{rh}^i(\partial_j \Gamma_{km}^{*i}) - \lambda_\ell H_{kr}^i(\partial_j \Gamma_{hm}^{*i}) - \lambda_\ell H_{rkh}^i P_{jm}^r + (\partial_j b_{\ell m}) H_{kh}^i \}. \end{aligned}$$

This shows that

$$H_{jpkh|\ell|m} = \lambda_\ell H_{jpkh|m} + b_{\ell m} H_{jpkh}$$

if and only if

$$\begin{aligned} (2.8) \quad & g_{ip} \left[\left\{ H_{kh}^r(\partial_j \Gamma_{r\ell}^{*i}) - H_{rh}^i(\partial_j \Gamma_{k\ell}^{*r}) - H_{kr}^i(\partial_j \Gamma_{h\ell}^{*r}) - H_{rkh}^i P_{j\ell}^r \right\}_m \right. \\ & + H_{kh|\ell}^r(\partial_j \Gamma_{rm}^{*i}) - H_{rh|\ell}^i(\partial_j \Gamma_{km}^{*i}) - H_{kr|\ell}^i(\partial_j \Gamma_{hm}^{*i}) - H_{kh|r}^i(\partial_j \Gamma_{\ell m}^{*r}) - \end{aligned}$$

$$\begin{aligned}
 & H_{rkh|\ell}^i P_{jm}^r - H_{kh}^s (\partial_r \Gamma_{s\ell}^{*i}) P_{jm}^r + H_{sh}^i (\partial_r \Gamma_{k\ell}^{*s}) P_{jm}^r + H_{ks}^i (\partial_r \Gamma_{h\ell}^{*s}) P_{jm}^r + \\
 & H_{skh}^i P_{r\ell}^s P_{jm}^r = \mathbf{g}_{ip} \{ (\partial_j \lambda_l) H_{kh|m}^i + \lambda_l H_{kh}^r (\partial_j \Gamma_{rm}^{*i}) - \lambda_l H_{rh}^i (\partial_j \Gamma_{km}^{*r}) \\
 & - \lambda_l H_{kr}^i (\partial_j \Gamma_{hm}^{*r}) - \lambda_l H_{rkh}^i P_{jm}^r + (\partial_j b_{lm}) H_{kh}^i \}.
 \end{aligned}$$

Thus, we conclude

Theorem 2.3. In K^h -GBR- F_n , the associate tensor H_{jpkh} of Berwald curvature tensor H_{jkh}^i is generalized birecurrent if and only if (2.8) holds good.

Contracting the indices i and h in (2.5), using (1.12) and (1.13), we get

$$\begin{aligned}
 (2.9) \quad & H_{jk|\ell|m} + \{ H_{kp}^r (\partial_j \Gamma_{r\ell}^{*p}) - H_r (\partial_j \Gamma_{k\ell}^{*r}) - H_{kr}^p (\partial_j \Gamma_{p\ell}^{*r}) - H_{rk} (\partial_j \Gamma_{j\ell}^{*r}) \}_{|m} \\
 & + H_{kp|\ell}^r (\partial_j \Gamma_{rm}^{*p}) - H_{r|\ell} (\partial_j \Gamma_{km}^{*r}) - H_{kr|\ell}^p (\partial_j \Gamma_{pm}^{*r}) - H_{k|r} (\partial_j \Gamma_{\ell m}^{*r}) - \\
 & H_{rk|\ell} P_{jm}^r - H_{kp}^s (\partial_r \Gamma_{s\ell}^{*p}) P_{jm}^r + H_s (\partial_r \Gamma_{k\ell}^{*s}) P_{jm}^r + H_{ks}^p (\partial_r \Gamma_{p\ell}^{*s}) P_{jm}^r + \\
 & H_{sk} P_{r\ell}^s P_{jm}^r = (\lambda_\ell H_{jk|m} + b_{\ell m} H_{jk}) + (\partial_j \lambda_\ell) H_{k|m} + \lambda_\ell H_{kp}^r (\partial_j \Gamma_{rm}^{*p}) \\
 & - \lambda_\ell H_r (\partial_j \Gamma_{km}^{*r}) - \lambda_\ell H_{kr}^p (\partial_j \Gamma_{pm}^{*r}) - \lambda_\ell H_{rk} P_{jm}^r + (\partial_j b_{\ell m}) H_k.
 \end{aligned}$$

This shows that

$$H_{jk|\ell|m} = \lambda_\ell H_{jk|m} + b_{\ell m} H_{jk}$$

if and only if

$$\begin{aligned}
 (2.10) \quad & \{ H_{kp}^r (\partial_j \Gamma_{r\ell}^{*p}) - H_r (\partial_j \Gamma_{k\ell}^{*r}) - H_{kr}^p (\partial_j \Gamma_{p\ell}^{*r}) - H_{rk} (\partial_j \Gamma_{j\ell}^{*r}) \}_{|m} \\
 & + H_{kp|\ell}^r (\partial_j \Gamma_{rm}^{*p}) - H_{r|\ell} (\partial_j \Gamma_{km}^{*r}) - H_{kr|\ell}^p (\partial_j \Gamma_{pm}^{*r}) - H_{k|r} (\partial_j \Gamma_{\ell m}^{*r}) - \\
 & H_{rk|\ell} P_{jm}^r - H_{kp}^s (\partial_r \Gamma_{s\ell}^{*p}) P_{jm}^r + H_s (\partial_r \Gamma_{k\ell}^{*s}) P_{jm}^r + H_{ks}^p (\partial_r \Gamma_{p\ell}^{*s}) P_{jm}^r + \\
 & H_{sk} P_{r\ell}^s P_{jm}^r = (\partial_j \lambda_\ell) H_{k|m} + \lambda_\ell H_{kp}^r (\partial_j \Gamma_{rm}^{*p}) - \lambda_\ell H_r (\partial_j \Gamma_{km}^{*r}) \\
 & \lambda_\ell H_{kr}^p (\partial_j \Gamma_{pm}^{*r}) - \lambda_\ell H_{rk} P_{jm}^r + (\partial_j b_{\ell m}) H_k.
 \end{aligned}$$

Theorem 2.4. In K^h -GBR- F_n , K -Ricci tensor H_{jk} in sense of Cartan is generalized birecurrent if and only if (2.10) holds good.

Contracting the indices i and j in (2.5) and using (1.16), we get

$$\begin{aligned}
 (2.11) \quad & (H_{hk} - H_{kh})_{|\ell|m} + \{H_{kh}^r(\dot{\partial}_p \Gamma_{rl}^{*p}) - H_{rh}^p(\dot{\partial}_p \Gamma_{kl}^{*r}) - H_{kr}^p(\dot{\partial}_p \Gamma_{hl}^{*r}) \\
 & - H_{rk}^p P_{p\ell}^r\}_{|m} + H_{kh|\ell}^r(\dot{\partial}_p \Gamma_{rm}^{*p}) - H_{rh|\ell}^p(\dot{\partial}_p \Gamma_{km}^{*r}) - H_{kr|\ell}^p(\dot{\partial}_p \Gamma_{hm}^{*r}) \\
 & - H_{kh|r}^p(\dot{\partial}_p \Gamma_{\ell m}^{*r}) - H_{rk|\ell}^p P_{pm}^r - H_{kh}^s(\dot{\partial}_r \Gamma_{s\ell}^{*p}) P_{pm}^r + H_{sh}^p(\dot{\partial}_r \Gamma_{k\ell}^{*s}) P_{pm}^r \\
 & + H_{ks}^p(\dot{\partial}_r \Gamma_{h\ell}^{*s}) P_{pm}^r + H_{skh}^p P_{r\ell}^s P_{pm}^r = \lambda_\ell (H_{hk} - H_{kh})_{|m} + \\
 & b_{\ell m} (H_{hk} - H_{kh}) + (\dot{\partial}_p \lambda_\ell) H_{kh|m}^p + \lambda_\ell H_{kh}^r(\dot{\partial}_p \Gamma_{rm}^{*p}) - \lambda_\ell H_{rh}^p(\dot{\partial}_p \Gamma_{km}^{*r}) \\
 & - \lambda_\ell H_{kr}^p(\dot{\partial}_p \Gamma_{hm}^{*r}) - \lambda_\ell H_{rk}^p P_{pm}^r + (\dot{\partial}_p b_{\ell m}) H_{kh}^p.
 \end{aligned}$$

This shows that

$$(H_{hk} - H_{kh})_{|\ell|m} = \lambda_\ell (H_{hk} - H_{kh})_{|m} + b_{\ell m} (H_{hk} - H_{kh})$$

if and only if

$$\begin{aligned}
 (2.12) \quad & \{H_{kh}^r(\dot{\partial}_p \Gamma_{rl}^{*p}) - H_{rh}^p(\dot{\partial}_p \Gamma_{kl}^{*r}) - H_{kr}^p(\dot{\partial}_p \Gamma_{hl}^{*r}) - H_{rk}^p P_{p\ell}^r\}_{|m} \\
 & + H_{kh|\ell}^r(\dot{\partial}_p \Gamma_{rm}^{*p}) - H_{rh|\ell}^p(\dot{\partial}_p \Gamma_{km}^{*r}) - H_{kr|\ell}^p(\dot{\partial}_p \Gamma_{hm}^{*r}) - H_{kh|r}^p(\dot{\partial}_p \Gamma_{\ell m}^{*r}) \\
 & - H_{rk|\ell}^p P_{pm}^r - H_{kh}^s(\dot{\partial}_r \Gamma_{s\ell}^{*p}) P_{pm}^r + H_{sh}^p(\dot{\partial}_r \Gamma_{k\ell}^{*s}) P_{pm}^r + H_{ks}^p(\dot{\partial}_r \Gamma_{h\ell}^{*s}) P_{pm}^r \\
 & + H_{skh}^p P_{r\ell}^s P_{pm}^r\} = b_{\ell m} (H_{hk} - H_{kh}) + \{(\dot{\partial}_p \lambda_\ell) H_{kh|m}^p + \lambda_\ell H_{kh}^r(\dot{\partial}_p \Gamma_{rm}^{*p}) \\
 & - \lambda_\ell H_{kr}^p(\dot{\partial}_p \Gamma_{hm}^{*r}) - \lambda_\ell H_{rk}^p P_{pm}^r + (\dot{\partial}_p b_{\ell m}) H_{kh}^p.
 \end{aligned}$$

Thus, we conclude

Theorem 2.5. In K^h -GBR- F_n , the tensor $(H_{hk} - H_{kh})$ is generalized birecurrent if and only if (2.12) holds good.

Differentiating (1.18) covariantly with respect to x^m in the sense of Cartan and using (1.1a), we get

$$\begin{aligned}
 (2.13) \quad & K_{jkh|\ell|m}^i + K_{j\ell k|h|m}^i + K_{jh\ell|k|m}^i + y^r \{(\dot{\partial}_s \Gamma_{jk}^{*i}) K_{r h \ell|m}^s + \\
 & (\dot{\partial}_s \Gamma_{j\ell}^{*i}) K_{r k h|m}^s + (\dot{\partial}_s \Gamma_{jh}^{*i}) K_{r \ell k|m}^s\} + y^r \{(\dot{\partial}_s \Gamma_{jk}^{*i})_{|m} K_{r h \ell}^s
 \end{aligned}$$

$$+(\partial_s \Gamma_{j\ell}^{*i})_{|m} K_{rkh}^s + (\partial_s \Gamma_{jh}^{*i})_{|m} K_{r\ell k}^s \} = 0.$$

Using (2.1) in (2.13), we get

$$(2.14) \quad \lambda_\ell K_{jkh|m}^i + \lambda_h K_{j\ell k|m}^i + \lambda_k K_{jh\ell|m}^i + b_{\ell m} K_{jkh}^i + b_{hm} K_{j\ell k}^i + b_{km} K_{jh\ell}^i \\ + y^r \{ (\partial_s \Gamma_{jk}^{*i}) K_{r\ell h}^s + (\partial_s \Gamma_{j\ell}^{*i}) K_{rkh}^s + (\partial_s \Gamma_{jh}^{*i}) K_{r\ell k}^s \} \\ + y^r \{ (\partial_s \Gamma_{jk}^{*i})_{|m} K_{r\ell h}^s + (\partial_s \Gamma_{j\ell}^{*i})_{|m} K_{rkh}^s + (\partial_s \Gamma_{jh}^{*i})_{|m} K_{r\ell k}^s \} = 0.$$

If Cartan's fourth curvature tensor K_{jkh}^i is recurrent which is given by (1.23), (2.14) becomes

$$(2.15) \quad \lambda_\ell \lambda_m K_{jkh}^i + \lambda_h \lambda_m K_{j\ell k}^i + \lambda_k \lambda_m K_{jh\ell}^i + b_{\ell m} K_{jkh}^i + b_{hm} K_{j\ell k}^i + b_{km} K_{jh\ell}^i \\ + \lambda_m y^r \{ (\partial_s \Gamma_{jk}^{*i}) K_{r\ell h}^s + (\partial_s \Gamma_{j\ell}^{*i}) K_{rkh}^s + (\partial_s \Gamma_{jh}^{*i}) K_{r\ell k}^s \} \\ + y^r \{ (\partial_s \Gamma_{jk}^{*i})_{|m} K_{r\ell h}^s + (\partial_s \Gamma_{j\ell}^{*i})_{|m} K_{rkh}^s + (\partial_s \Gamma_{jh}^{*i})_{|m} K_{r\ell k}^s \} = 0$$

Putting (1.18) in (2.15), we get K_{jkh}^i

$$\lambda_\ell \lambda_m K_{jkh}^i + \lambda_h \lambda_m K_{j\ell k}^i + \lambda_k \lambda_m K_{jh\ell}^i + b_{\ell m} K_{jkh}^i \\ + b_{hm} K_{j\ell k}^i + b_{km} K_{jh\ell}^i - \lambda_m (K_{jkh|\ell}^i + K_{j\ell k|h}^i + K_{jh\ell|k}^i) \\ + y^r \{ (\partial_s \Gamma_{jk}^{*i})_{|m} K_{r\ell h}^s + (\partial_s \Gamma_{j\ell}^{*i})_{|m} K_{rkh}^s + (\partial_s \Gamma_{jh}^{*i})_{|m} K_{r\ell k}^s \} = 0$$

which can be written as

$$(2.16) \quad b_{\ell m} K_{jkh}^i + b_{hm} K_{j\ell k}^i + b_{km} K_{jh\ell}^i + y^r \{ (\partial_s \Gamma_{jk}^{*i})_{|m} K_{r\ell h}^s + \\ + (\partial_s \Gamma_{j\ell}^{*i})_{|m} K_{rkh}^s + (\partial_s \Gamma_{jh}^{*i})_{|m} K_{r\ell k}^s \} = 0.$$

Transvecting (2.16) by y^j , using (1.1a), (1.6) and (1.2b), we get

$$b_{\ell m} H_{kh}^i + b_{hm} H_{\ell k}^i + b_{km} H_{h\ell}^i + P_{sk|m}^i H_{h\ell}^s + P_{s\ell|m}^i H_{kh}^s + P_{sh|m}^i H_{\ell k}^s = 0.$$

3. K^h - Generalized Birecurrent Affinely Connected Space

Let us consider an affinely connected or Berwald's space which is characterized by any one of the equivalent conditions (1.19), (1.20), (1.21) and (1.22).

Definition 3.1. The K^h -generalized birecurrent space is called K^h -generalized birecurrent affinely connected if it satisfies any one of the conditions (1.19), (1.20), (1.21) and (1.22) and denoted briefly by K^h -GBR-affinely connected space.

Let us consider K^h -GBR-affinely connected space.

If $\partial_j \lambda_\ell = 0$, $\partial_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.5) reduces to

$$(1.3) \quad H_{jkh|\ell|m}^i = \lambda_\ell H_{jkh|m}^i + b_{\ell m} H_{jkh}^i.$$

Thus, we conclude

Theorem 3.1. In K^h -GBR-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then Berwald curvature tensor H_{jkh}^i is generalized birecurrent.

If $\partial_j \lambda_\ell = 0$, $\partial_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.7) reduces to

$$H_{jpkh|\ell|m} = \lambda_\ell H_{jpkh|m} + b_{\ell m} H_{jpkh}.$$

Thus, we conclude

Theorem 3.2. In K^h -GBR-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the associate tensor H_{jpkh} of Berwald curvature tensor H_{jkh}^i is generalized birecurrent.

If $\partial_j \lambda_\ell = 0$, $\partial_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.9) reduces to

$$H_{jk|\ell|m} = \lambda_\ell H_{jk|m} + b_{\ell m} H_{jk}.$$

Thus, we conclude

Theorem 3.3. In K^h -GBR-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the Ricci tensor H_{jk} in sense of Berwald is generalized birecurrent.

If $\partial_j \lambda_\ell = 0$, $\partial_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.11) reduces to

$$(H_{hk} - H_{kh})_{|\ell|m} = \lambda_\ell (H_{hk} - H_{kh})_{|m} + b_{\ell m} (H_{hk} - H_{kh}).$$

Thus, we conclude

Theorem 3.4. In $K^h - GBR$ –affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the tensor $(H_{hk} - H_{kh})$ is generalized birecurrent.

Now, transvecting (3.1) by y^j , using (1.1a) and (1.8), we get

$$(3.2) \quad H_{kh|\ell|m}^i = \lambda_\ell H_{kh|m}^i + b_{\ell m} H_{kh}^i.$$

Transvecting (3.2) by y^k , using (1.1a) and (1.11), we get

$$(3.3) \quad H_{h|\ell|m}^i = \lambda_\ell H_{h|m}^i + b_{\ell m} H_h^i.$$

Contracting the indices i and h in (3.2) and using (1.13), we get

$$H_{k|\ell|m} = \lambda_\ell H_{k|m} + b_{\ell m} H_k$$

Contracting the indices i and h in (3.3) and using (1.14), we get

$$H_{|\ell|m} = \lambda_\ell H_{|m} + b_{\ell m} H.$$

Transvecting (3.2) by g_{ip} , using (1.1b) and (1.17), we get

$$H_{kp.h|\ell|m} = \lambda_\ell H_{kp.h|m} + b_{\ell m} H_{kp.h}.$$

Thus, we conclude

Theorem 3.5. In $K^h - GBR$ –affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the $h(v)$ – torsion tensor H_{kh}^i , the deviation tensor H_h^i , the curvature vector H_k , the curvature scalar H and the tensor $H_{kp.h}$ are all generalized birecurrent.

In view of (1.22), the equation (2.14) can be written as

$$(3.4) \quad \lambda_\ell K_{jkh|m}^i + \lambda_h K_{j\ell k|m}^i + \lambda_k K_{jh\ell|m}^i + b_{\ell m} K_{jkh}^i + b_{hm} K_{j\ell k}^i + b_{km} K_{jh\ell}^i = 0.$$

In view of (1.23), (3.4) reduces to

$$(\lambda_\ell \lambda_m + b_{lm}) K_{jkh}^i + (\lambda_h \lambda_m + b_{hm}) K_{j\ell k}^i + (\lambda_k \lambda_m + b_{km}) K_{jh\ell}^i = 0$$

which can be written as

$$(3.5) \quad a_{\ell m} K_{jkh}^i + a_{hm} K_{j\ell k}^i + a_{km} K_{jh\ell}^i = 0,$$

where $a_{rm} = \lambda_r \lambda_m + b_{rm}$ is covariant tensor field of second order.

Transvecting (3.5) by y^j , using (1.1a) and (1.6), we get

$$(3.6) \quad a_{\ell m} H_{kh}^i + a_{hm} H_{\ell k}^i + a_{km} H_{h\ell}^i = 0.$$

Thus, we conclude

Theorem 3.6. *In K^h -GBR-affinely connected space, the identities (3.5) and (3.6) are hold good.*

Contracting the indices i and ℓ in (3.5), using (1.3) and (1.5), we get

$$(3.7) \quad a_{pm} K_{jkh}^p - a_{hm} K_{jk} + a_{km} K_{jh} = 0$$

which can be written as

$$(3.8) \quad K_{jkh}^p = \frac{(a_{hm} K_{jk} - a_{km} K_{jh})}{a_{pm}}.$$

Thus, we conclude

Theorem 3.7. *In K^h -GBR-affinely connected space, Cartan's fourth curvature tensor K_{jkh}^p is defined by (3.8).*

In view of (1.2b) and (1.22), (1.7) reduces to

$$(3.9) \quad H_{jkh}^i = K_{jkh}^i.$$

Putting (3.9) in (3.5), we get

$$(3.10) \quad a_{\ell m} H_{jkh}^i + a_{hm} H_{j\ell k}^i + a_{km} H_{jh\ell}^i = 0.$$

Thus, we conclude

Theorem 3.8. *In K^h -GBR-affinely connected space, Berwald curvature tensor H_{jkh}^i coincide with Cartan's fourth curvature tensor K_{jkh}^i and the identity (3.10) holds good.*

Contracting the indices i and ℓ in (3.10), using (1.12), (1.10b) and the skew-symmetric property of Berwald curvature tensor H_{jkh}^i in it's last two lower indices, we get

$$a_{pm} H_{jkh}^p - a_{hm} H_{jk} + a_{km} H_{jh} = 0.$$

which can be written as

$$(3.11) \quad H_{jkh}^p = \frac{(a_{hm} H_{jk} - a_{km} H_{jh})}{a_{pm}}.$$

Thus, we conclude

Theorem 3.9. *In K^h -GBR-affinely connected space, Berwald curvature tensor is defined by (3.11).*

Contracting the indices i and h in (3.9), using (1.12) and (1.5), we get

$$H_{jk} = K_{jk}.$$

Thus, we conclude

Theorem 3.10. *In K^h -GBR-affinely connected space, Ricci tensor H_{jk} in sense of Berwald coincide with Ricci tensor K_{jk} of Cartan's fourth curvature.*

Transvecting (3.9) by g_{ip} , using (1.1b), (1.15) and (1.4), we get

$$H_{jpkh} = K_{jpkh}.$$

Thus, we conclude

Theorem 3.11. *In K^h -GBR-affinely connected space, the associate curvature tensor H_{jpkh} of Berwald curvature tensor coincide with the associate curvature tensor K_{jpkh} of Cartan's fourth curvature tensor.*

4. Conclusion

(3.1) The K^h -generalized birecurrent space is called K^h -generalized birecurrent affinely connected if it satisfies any one of the conditions (1.19), (1.20), (1.21) and (1.22).

(3.2) In K^h -GBR-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then Berwald curvature tensor H_{jkh}^i is generalized birecurrent.

(3.3) In K^h -GBR-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the $h(v)$ -torsion tensor H_{kh}^i , the deviation tensor H_h^i , the curvature vector H_k , the curvature scalar H and the tensor $H_{kp,h}$ are all generalized birecurrent.

(3.4) In K^h -GBR-affinely connected space, Cartan's fourth curvature tensor K_{jkh}^p is defined by (3.8).

(3.5) In K^h -GBR-affinely connected space, Berwald curvature tensor H_{jkh}^i coincide with Cartan's fourth curvature tensor K_{jkh}^i and the identity (3.10) holds good.

(3.6) In K^h -GBR-affinely connected space, Ricci tensor H_{jk} in sense of Berwald coincide with Ricci

tensor K_{jk} of Cartan's fourth curvature.

(3.7) In K^h -GBR-affinely connected space, the associate curvature tensor H_{jpkh} of Berwald

curvature tensor coincide with the associate curvature tensor K_{jpkh} of Cartan's fourth curvature

tensor.

5. Recommendations

Authors recommend the need for the continuing research and development in affinely connected space and its relation with other spaces.

References

- [1] A. A. A. Muhib. "On independent components of tensor, I- relative tensor and R^h - generalized trirecurrent Finsler space", M. Sc. Thesis, University of Aden, Aden, Yemen, 2009.
- [2] A. A. M. Saleem. "On Generalized Birecurrent and Trirecurrent Finsler Spaces", M. Sc. Thesis, University of Aden, Aden, Yemen, 2011.
- [3] H. Rund: The differential geometry of Finsler space, Springer - Verlag, Berlin Gottingen-Heidelberg, 1959; 2nd Edit. (in Russian), Nauka, (Moscow), 1981.
- [4] H.D. Pande and B. Single. "On existence of the affinely connected Finsler spaces with recurrent tensor field, Reprinted from India Journal Pure and Applied of Mathematics, Vol. 8, No. 3,(March 1977), pp. 295-301.
- [5] M. A. A. Ali "On K^h - birecurrent Finsler space." M.Sc. Thesis, University of Aden, Aden, Yemen, 2014.
- [6] N.S.H. Hussien "On K^h - recurrent Finsler space." M.sc. Thesis, University of Aden, Aden, Yemen,2014.
- [7] P. K. Dwivedi: P^* - Reducible Finsler spaces and application, Int. Journal of Math. Analysis, Vol. 5, No. 5, 2011, pp. 223-229.